Scaling groups

I. Definitions / first properties

Def: let (X, d_X), (Y, d_Y) metric spaces.
A map $f: X \to Y$ is a Q.I. if there are constants $C \geq 1$, $K \geq 0$ s.t.

(i) $$\frac{1}{C} d_X(x,y) - K \leq d_Y(f(x), f(y)) \leq C \cdot d_X(x,y) + K \quad \forall x,y \in X$$

(ii) $d_Y(y, f(X)) \leq K \quad \forall y \in Y$.

Equivalently, f is a Q.I $\Leftrightarrow$ (i) holds and $\exists g: Y \to X$ s.t.

$$d(g \circ f, \mathrm{Id}_X), \; d(f \circ g, \mathrm{Id}_Y) \leq K.$$

where for $h_1, h_2 : X \to Y$:

$$d(h_1, h_2) := \sup_{x \in X} d_Y(h_1(x), h_2(x)).$$

Observation: a Q.I. $f: X \to Y$ has uniformly bounded fibers: $\exists P \geq 1$, $|f^{-1}(\{y\})| \leq P \ \forall y \in Y$.

If $x, x' \in f^{-1}(\{y\})$, then

$$d_X(x,x') \leq C \cdot \big(d(f(x), f(x')) + K \big) = CK$$

<u>Def</u>: $k > 0$. A Q.I. $f: X \to Y$ is quasi-k-to-one if $\exists C > 0$:

$$\forall A \subset Y \text{ finite} \quad \big| k|A| - |f^{-1}(A)| \big| \leq C \cdot |\partial_Y A|$$

where $\partial_Y A := \{ y \in Y \setminus A : \exists a \in A,\ y \underset{Y}{\sim} a \}$.

<u>Theorem</u> (Whyte, 1999): X, Y graphs.
A Q.I. $f: X \to Y$ is quasi-one-to-one if and only if f is at bounded distance from a bijection.

Between non amenable spaces, any Q.I. is quasi-k-to-one for any $k > 0$:

Y not amenable $\Rightarrow \exists \varepsilon > 0$, $|\partial_Y A| > \varepsilon \cdot |A|$ $\forall A \subset Y$ finite

$$\big|\, k|A| - |f^{-1}(A)| \,\big| \le k\cdot|A| + \underbrace{|f^{-1}(A)|}_{\le P\cdot|A|}$$

$$\le (k+P)\cdot|A|$$

$$\le \frac{k+P}{\varepsilon}\cdot|\partial_Y A|.$$

Lemma: X, Y amenable graphs.
If $f: X \to Y$ is quasi-k-to-one and quasi-k'-to-one, then $k = k'$.

Proof: $\big|\, k|A| - |f^{-1}(A)| \,\big| \le C\cdot|\partial A| \quad \forall A$ finite

Apply to $(F_n)_{n\in\mathbb{N}}$ a Følner sequence of Y:

$$\left|\, k - \frac{|f^{-1}(F_n)|}{|F_n|} \,\right| \le C\cdot\frac{|\partial F_n|}{|F_n|}$$

$$\overset{n\to\infty}{\Longrightarrow} \quad k = \lim_{n\to\infty} \frac{|f^{-1}(F_n)|}{|F_n|} = k'. \quad \square$$

Proposition: $k_1, k_2 > 0$. $f: X \to Y$, $g: Y \to Z$, $h: X \to Y$.

(i) f quasi-k_1-to-one and h is at bounded distance from f, then h is quasi-k_1-to-one.

(ii) If f is quasi-k_1-to-one, g is quasi-k_2-to-one, then $g \circ f$ is quasi-$k_1 k_2$-to-one.

(iii) f quasi-k_1-to-one, then any quasi-inverse of f is quasi-$\frac{1}{k_1}$-to-one.

Lemma: X bounded degree graph.

(i) $\forall A \subset X$ finite, $S \geq 0$, then

$$|A^{+S}| \leq N^S \cdot |A|$$

where $N \geq 1$ is a fixed upper bound on degree of vertices of X

$A^{+S} := \{ b \in X : \exists a \in A, d(a,b) \leq S \}$

$$= \bigcup_{a \in A} B(a, S).$$

(ii) $\forall A \subset X$ finite, $S \geq 0$, $\exists R > 0$ s.t.

$$|A^{+S} \setminus A| \leq R \cdot |\partial_X A|.$$

Proof: (i) If $a \in A$, there are $\leq N^{i}$ paths starting from a and ending at an element of A^{+i}:

$$|A^{+S}| \leq |A| \cdot \sum_{i=0}^{S-1} N^{i} \leq N^{S} \cdot |A|.$$

(ii) $A^{+S} \setminus A \subseteq (\partial_X A)^{+(S-1)}$

and apply (i). $\square$

Lemme: $f : X \to Y$ Q.I.

$\exists L > 0$, $|\partial_X f^{-1}(A)| \leq L \cdot |\partial_Y A|$ $\forall A \subset Y$ finite.

Proof: Let $x \in \partial_X f^{-1}(A)$. Then $\exists y \in f^{-1}(A)$ with $d_X(x, y) = 1$.

Then $d_Y(f(x), \underbrace{f(y)}_{\in A}) \leq C \cdot d_X(x, y) + K = C + K$

$\Rightarrow \quad f(x) \in A^{+(C+K)}$

$\Rightarrow \quad x \in f^{-1}(A^{+(C+K)})$. Thus

$$
\begin{aligned}
|\partial_X f^{-1}(A)| &\leq |f^{-1}(A^{+(C+K)}) \setminus f^{-1}(A)| \\
&= |f^{-1}(A^{+(C+K)} \setminus A)| \\
&\leq P \cdot |A^{+(C+K)} \setminus A|, \quad P \text{ uniform bound on } |f^{-1}(\{y\})| \\
&\leq P \cdot R \cdot |\partial_Y A|. \qquad \square
\end{aligned}
$$

Proof of the Proposition:

(i) $d(f, h) \leq Q$.

This implies that $f^{-1}(B) \subset h^{-1}(B^{+Q})$

$h^{-1}(B) \subset f^{-1}(B^{+Q})$.

Let $A \subset Y$ finite:

$$
\begin{aligned}
\big| k_1 |A| - |h^{-1}(A)| \big| &\leq \big| k_1 |A| - |f^{-1}(A)| \big| \\
&\quad + \big| |f^{-1}(A)| - |h^{-1}(A)| \big| \\
&\leq C \cdot |\partial A| + \big| |f^{-1}(A)| - |h^{-1}(A)| \big|
\end{aligned}
$$

and:

$$|f^{-1}(A)| - |h^{-1}(A)| \leq |h^{-1}(A^{+Q})| - |h^{-1}(A)|$$

$$= |h^{-1}(A^{+Q} \setminus A)|$$

$$\leq P \cdot |A^{+Q} \setminus A| \quad , \quad P \text{ uniform bound on } |h^{-1}(\{y\})|$$

$$\leq P \cdot R \cdot |\partial A|$$

$$|h^{-1}(A)| - |f^{-1}(A)| \leq |f^{-1}(A^{+Q} \setminus A)|$$

$$\leq P' \cdot R \cdot |\partial A| \quad , \quad P' \text{ uniform bound on } |f^{-1}(\{y\})|$$

$$\Rightarrow \big| k_1 |A| - |h^{-1}(A)| \big| \leq \big(C + \max(P, P') \cdot R \big) \cdot |\partial A|$$

$\Rightarrow$ h is quasi-k_1-to-one.

(ii) $A \subset \mathbb{Z}$ finite: $\quad C \rightsquigarrow$ constant for g

$D \rightsquigarrow$ constant for f

$$\big| k_1 k_2 |A| - |(g \circ f)^{-1}(A)| \big|$$

$$\leq \big| k_1 k_2 |A| - k_1 |g^{-1}(A)| \big| + \big| k_1 |g^{-1}(A)| - |f^{-1}(g^{-1}(A))| \big|$$

$$\leq k_1 \big| k_2 |A| - |g^{-1}(A)| \big| + D \cdot |\partial_Y g^{-1}(A)|$$

$$\leq k_1 \cdot C \cdot |\partial_2 A| + D \cdot L \cdot |\partial_2 A|$$

$$= (k_1 \cdot C + D \cdot L) |\partial_2 A| .$$

$\Rightarrow$ $g \circ f$ is quasi-$k_1 k_2$-to-one. $\square$

Corollary: $H_1, H_2 \leq_f G$.

$\exists$ a quasi-$\dfrac{[G:H_2]}{[G:H_1]}$-to-one Q.I.

$$H_1 \longrightarrow H_2 .$$

Proof: $H \leq_f G$, then $H \hookrightarrow G$ is quasi-$\frac{1}{[G:H]}$-to-one.

$$\begin{array}{ccc} & G & \\ {\scriptstyle \frac{1}{[G:H_1]}} \nearrow & & \downarrow {\scriptstyle [G:H_2]} \\ H_1 & & H_2 \end{array}$$

Corollary: $H_1, H_2 \leq_f G$.

If $[G:H_1] = [G:H_2]$, then H_1 and H_2 are biLipschitz equivalent

(i.e. bijective quasi-isometries,

i.e. bijective Lipschitz maps $H_1 \to H_2$ whose inverse is also Lipschitz.)

X amenable, bounded degree graph:

$$Sc : QI_{sc}(X) \longrightarrow \mathbb{R}_{>0}$$
$$f \text{ quasi-}k\text{-to-one} \mapsto k$$

is a group morphism. Then

$$Sc(X) := \operatorname{Im}(Sc) \text{ is a}$$

subgroup of $(\mathbb{R}_{>0}, \cdot)$.

Questions: $X = Cay(G,S)$

- Is it true that any Q.I. $G \to G$ is scaling?
- In general, what is $Sc(G)$?

Proposition: $Sc(G) = \mathbb{R}_{>0}$ if:

(i) $G = BS(1,n)$, $n \geq 2$

(ii) Carnot groups (e.g. $\mathbb{Z}^d$, $Heis(\mathbb{Z})$)

(iii) $G = SOL(\mathbb{R})$ or any of its lattices.

Proposition: let F be a finite group. (Tullia Dymarz, 2010)

Then one has

$$Sc(F \wr \mathbb{Z}) = \langle p_1, \dots, p_k \rangle$$

where $p_1, \dots, p_k$ are the prime numbers in the decomposition of $|F|$.

Algebraic consequences:

Corollary: G f.g. amenable. If $Sc(G) = \{1\}$, then G has no f.i. proper subgroup isomorphic to itself.

Proof: If $H \leq_{f.i.} G$, $[G:H] < \infty$

$$\begin{matrix} H \\ \cong \\ G \end{matrix}$$

$$\Rightarrow \frac{1}{[G:H]} \in Sc(G) = \{1\} \Rightarrow [G:H] = 1 \Rightarrow H = G. \quad \square$$

Corollary: G f.g. amenable. If $Sc(G) = \{1\}$, if $H_1, H_2 \leq_f G$ are biLipschitz equivalent, then $[G:H_1] = [G:H_2]$.

Lamplighters

Def: A, B two groups.

$$A \wr B := \left(\bigoplus_B A\right) \rtimes B$$

$$(b \cdot f)(b') := f(b^{-1}b')$$

If $A = \langle \{a_1, \dots, a_n\} \rangle$, $B = \langle \{b_1, \dots, b_m\} \rangle$, then $A \wr B$ is generated by "T"

$$U := \{(\delta_{a_i}, 1_B), (\mathbb{1}, b_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$$

$\mathbb{1} : B \to A, \; b \mapsto 1_A$

$\delta_{a_i} : B \to A, \; b \mapsto \begin{cases} a_i & \text{if } b = 1_B \\ 1_A & \text{otherwise.} \end{cases}$

$(c, p) \in A \wr B$: c colouring of $Cay(B, T)$
$p \in B$ an arrow moving in $Cay(B, T)$.

2 elementary moves in $Cay(A \wr B, U)$:

- either c stays the same, and the arrow goes from p to $p b_j$ (use $(\mathbb{1}, b_j)$)

- or the arrow stays where it stands, and change the color at p, that goes from $c(p)$ to $c(p)a_i$ (use $(\delta_{a_i}, 1_B)$).

<u>Theorem</u> (Genevois – Tessera, 2024):
Let F_1, F_2 be non trivial finite groups. G, H amenable, finitely presented and one-ended groups. Then there exist $a, r, s \geq 1$ such that any quasi-isometry $F_1 \wr G \to F_2 \wr H$ is quasi-$\frac{s}{r}$-to-one, and if there is indeed such a Q.I., $|F_1| = a^r$, $|F_2| = a^s$.

<u>Corollary</u>: F finite, $\neq \{1\}$.
G amenable, f.p., one ended.

Then any Q.I. $F \wr G \to F \wr G$ is at bounded distance from a bijection. In particular, $Sc(F \wr G) = \{1\}$.

Theorem: F_1, F_2 finite groups, $\neq \{1\}$, G, H amenable f.p., one-ended. Let $\varphi : F_1 \wr G \longrightarrow F_2 \wr H$ be a Q.I. Then there exist a bijection

$$\alpha : \bigoplus_G F_1 \longrightarrow \bigoplus_H F_2 ,$$

a Q.I. $\beta : G \to H$ such that φ is at bounded distance from

$$\psi : F_1 \wr G \longrightarrow F_2 \wr H$$
$$(c, p) \mapsto (\alpha(c), \beta(p))$$

and any quasi-inverse of φ is at bounded distance from

$$\overline{\psi} : F_2 \wr H \longrightarrow F_1 \wr G$$
$$(c, p) \mapsto (\alpha^{-1}(c), \overline{\beta}(p))$$

$\overline{\beta} : H \longrightarrow G$ is a quasi-inverse of β.

Lemma: Let $\alpha : \bigoplus_G F_1 \longrightarrow \bigoplus_H F_2$

$\beta : G \longrightarrow H$ s.t.

$$\varphi : F_1 \wr G \longrightarrow F_2 \wr H$$

$$(c, p) \mapsto (\alpha(c), \beta(p))$$

is a Q.I. If β is quasi-k-to-one then φ is quasi-k-to-one.

Proof: Fix $A \subset F_2 \wr H$ finite.

Let $\mathcal{C} \subset \bigoplus_G F_2$ be the collection of colourings that appear as the first coordinate of an element of A, and for $c \in \mathcal{C}$, let $A_c \subset A$ be the set of elements of A having c as first coordinate. Thus

$$A = \bigsqcup_{c \in \mathcal{C}} A_c$$

$$\big|k|A| - |\varphi^{-1}(A)|\big| \leq \sum_{c \in \mathcal{C}} \big|k \cdot |A_c| - |\varphi^{-1}(A_c)|\big|$$

$$\overset{(*)}{=} \sum_{c \in \mathcal{C}} \big|k \cdot |B_c| - |\beta^{-1}(B_c)|\big|$$

where $B_c := \pi_H(A_c)$, since

$$\begin{tikzcd}
F_1 \wr G \arrow[r, mapsto, "\varphi"] \arrow[d, "\pi_G"] & F_2 \wr H \supset A \supset A_c \arrow[d, "\pi_H"] & \arrow[d] \\
G \arrow[r, "G", "\beta"'] & H & B_c
\end{tikzcd}$$

$$(*) \leq \sum_{c \in \mathcal{C}} C \cdot |\partial_H B_c|$$

$$\leq C \cdot |\partial_{F_2 \wr H} A|$$

because $\partial_{F_2 \wr H} A$ contains $\bigsqcup_{c \in \mathcal{C}} \{c\} \times \partial B_c$

$\Longrightarrow \varphi$ is quasi-k-to-one. $\square$

Proposition: Notations as above.

$$\varphi : F_1 \wr G \longrightarrow F_2 \wr H \qquad \text{Q.I.}$$
$$(c,p) \longmapsto (\alpha(c), \beta(p))$$

Then $\exists\, Q \geq 0$ such that

$$d_{Haus}\Big(\beta(\mathrm{supp}(c_1^{-1}c_2)),\ \mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))\Big) \leq Q$$

for all $c_1, c_2 \in \bigoplus_G F_1$.

Notations: in (X, d_X), $A \subset X$

$$A^{+R} := \bigcup_{a \in A} B_{d_X}(a, R)$$

$$d_{Haus}(A,B) := \inf\{R \geq 0 : A \subset B^{+R},\ B \subset A^{+R}\}.$$

Proof: Let $C \geq 1$, $K \geq 0$ be the parameters of φ, and of

$$\overline{\varphi} : F_2 \wr H \longrightarrow F_1 \wr G$$
$$(c,p) \longmapsto (\alpha^{-1}(c), \overline{\beta}(p)).$$

Fix $c_1, c_2 \in \bigoplus_G F_1$. Fix a sequence of colourings

$$a_1 = c_1, a_2, \dots, a_n = c_2$$

such that a_i, a_{i+1} differ only on one point p_i, $\forall 1 \leq i \leq n-1$.

This way, $\mathrm{supp}(c_1^{-1} c_2) = \{p_1, \dots, p_{n-1}\}$.

We have:

$$d\Big((\alpha(a_i), \beta(p_i)), (\alpha(a_{i+1}), \beta(p_i)) \Big)$$

$$= d\big(\varphi(a_i, p_i), \varphi(a_{i+1}, p_i) \big)$$

$$\leq C \cdot d\big((a_i, p_i), (a_{i+1}, p_i) \big) + K$$

(c_1, p_1)
(c_2, p_2)

$$= C + K$$

$\Rightarrow \alpha(a_i), \alpha(a_{i+1})$ can only differ on $B_H(\beta(p_i), C+K)$ $\forall 1 \leq i \leq n-1$

$\Rightarrow \alpha(c_1) = \alpha(a_1),\ \alpha(c_2) = \alpha(a_n)$

can only differ

$$\bigcup_{i=1}^{n-1} B_H(\beta(p_i), C+k) = \bigcup_{i=1}^{n-1} \{\beta(p_i)\}^{+(C+k)}$$

$$= \{\beta(p_1), \dots, \beta(p_{n-1})\}^{+(C+k)}$$

$$= \beta(\mathrm{supp}(c_1^{-1}c_2))^{+(C+k)}$$

i.e.:

$$\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)) \subset \beta(\mathrm{supp}(c_1^{-1}c_2))^{+(C+k)}.$$

By symmetry,

$\forall d_1, d_2 \in \bigoplus_H F_2$:

$$\mathrm{supp}(\alpha^{-1}(d_1)^{-1}\alpha(d_2)) \subset \overline{\beta}(\mathrm{supp}(d_1^{-1}d_2))^{+(C+k)}$$

$d_1 = \alpha(c_1),\ d_2 = \alpha(c_2)$:

$$\mathrm{supp}(c_1^{-1}c_2) \subset \overline{\beta}(\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)))^{+(C+k)}$$

$\Downarrow$ apply β

$$\beta(\mathrm{supp}(c_1^{-1}c_2)) \subset \beta\left(\bar{\beta}(\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)))^{+(C+K)}\right)$$

$$\subset \beta\left(\bar{\beta}(\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)))\right)^{+(C(C+K)+K)}$$

$$\subset \mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))^{+(C(C+K)+K+K)}$$

$\Longrightarrow$

$$d_{Haus}\left(\beta(\mathrm{supp}(c_1^{-1}c_2)), \mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))\right)$$

$$\leq C(C+K) + 2K =: Q. \qquad \square$$

Lemma 1: Let $n, m \geq 2$. Let G, H infinite groups. Let

$$\varphi : \mathbb{Z}/n\mathbb{Z} \wr G \longrightarrow \mathbb{Z}/m\mathbb{Z} \wr H$$

$$(c, p) \longmapsto (\alpha(c), \beta(p))$$

for some bijection α and Q.I. $\beta : G \to H$.

For any quasi-inverse $\bar{\beta}$ of β, there exists $Q \geq 0$ s.t:

- For all $A_1 \subset G$, all $Q' \geq Q$,

  $\alpha^{-1}\big(\underbrace{\mathcal{L}(\beta(A_1)^{+Q'})}_{\text{colourings supported on } \beta(A_1)^{+Q'}}\big)$ is a union of cosets of $\mathcal{L}(A_1)$

- For all $A_2 \subset H$, $Q' \geq Q$,

  $\alpha\big(\mathcal{L}(\bar{\beta}(A_2)^{+Q'})\big)$ is a union of cosets of $\mathcal{L}(A_2)$.

As a consequence, n and m have the same prime divisors.

Proof: $A \subset G$ finite

$$\alpha^{-1}\big(\mathcal{L}(\beta(A)^{+Q'})\big) = \bigsqcup_{i=1}^{k} d_i \, \mathcal{L}(A)$$

for some $k \geq 1$, $d_1, \dots, d_k \in \bigoplus_H \mathbb{Z}/m\mathbb{Z}$.

$$m^{|\beta(A)^{+Q'}|} = |\mathcal{X}(\beta(A)^{+Q'})|$$

$$= |\alpha^{-1}(\mathcal{X}(\beta(A)^{+Q'}))|$$

$$= |\bigsqcup_{i=1}^{k} d_i\, \mathcal{X}(A)| = k \cdot |\mathcal{X}(A)| = k \cdot n^{|A|}.$$

$\Rightarrow n, m$ have same prime divisors. $\square$

<u>Recall</u>: In $\mathbb{Z}/n\mathbb{Z} \wr G$, for $A \subset G$, $\mathcal{X}(A) \subset \bigoplus_G \mathbb{Z}/n\mathbb{Z}$ stands for the subgroup made of colourings that are supported only on A.

<u>Lemma 2</u>: Let $n, m \geq 2$, G, H infinite groups. Assume $\exists Q \geq 0$ and two maps $\alpha : \bigoplus_G \mathbb{Z}/n\mathbb{Z} \longrightarrow \bigoplus_H \mathbb{Z}/m\mathbb{Z}$ such that, for any colourings $c_1, c_2 \in \bigoplus_G \mathbb{Z}/n\mathbb{Z}$, if $\mathrm{supp}(c_1^{-1}c_2) \subset \{p\}$, then

$$\mathrm{supp}\left(\alpha(c_1)^{-1}\alpha(c_2)\right) \subset B_H(\beta(p), Q).$$

Then, for any finite $A \subset G$, any $c \in \bigoplus_G \mathbb{Z}/n\mathbb{Z}$, we have

$$\alpha\left(c\,\mathcal{Z}(A)\right) \subset \alpha(c)\,\mathcal{Z}\left(\beta(A)^{+Q}\right).$$

Proof of Lemma 2: By induction over $|A|$.

If $|A| = 1$, and let $c \in \bigoplus_G \mathbb{Z}/n\mathbb{Z}$.

Then, if $d \in c\,\mathcal{Z}(A)$, then $c^{-1}d \in \mathcal{Z}(A)$

i.e. $\mathrm{supp}(c^{-1}d) \subset \{p\}$, $p \in G$

$$\underset{\text{hypothesis}}{\Longrightarrow} \quad \mathrm{supp}\left(\alpha(c)^{-1}\alpha(d)\right) \subset B_H(\beta(p), Q)$$

i.e. $\alpha(c)^{-1}\alpha(d) \in \mathcal{Z}\left(\{\beta(p)\}^{+Q}\right)$

i.e. $\alpha(d) \in \alpha(c)\,\mathcal{Z}\left(\beta(A)^{+Q}\right)$.

Assume that the claim holds for a given cardinality and for every colouring. Let $a \in A$, $c' \in c\,\mathcal{Z}(A)$.

Then there is $c'' \in c\,\mathcal{L}(A\setminus\{a\})$ such that $c' \in c''\,\mathcal{L}(\{a\})$. By the inductive assumption:

$$\alpha(c'') \in \alpha(c)\,\mathcal{L}\left(\beta(A\setminus\{a\})^{+Q}\right)$$

$$\alpha(c') \in \alpha(c'')\,\mathcal{L}\left(\beta(a)^{+Q}\right).$$

Thus:

$$\alpha(c') \in \alpha(c'')\,\mathcal{L}(\beta(a)^{+Q})$$

$$\subset \alpha(c)\,\mathcal{L}(\beta(A\setminus\{a\})^{+Q})\,\mathcal{L}(\beta(a)^{+Q})$$

$$= \alpha(c)\,\mathcal{L}\left(\beta(A\setminus\{a\})^{+Q} \cup \beta(a)^{+Q}\right)$$

$$= \alpha(c)\,\mathcal{L}\left(\beta(A)^{+Q}\right). \qquad \square$$

Proof of Lemma 1: φ is aptolic

$$\varphi : \mathbb{Z}/n\mathbb{Z} \wr G \longrightarrow \mathbb{Z}/m\mathbb{Z} \wr H$$

$$(c, p) \longmapsto (\alpha(c), \beta(p))$$

By the proposition of last week, $\exists Q \geq 0$ s.t:

$$d_{Haus}\left(\beta(\mathrm{supp}(c_1^{-1}c_2)),\ \mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))\right) \le Q$$

for all $c_1, c_2 \in \bigoplus_G \mathbb{Z}/n\mathbb{Z}$. In particular, the assumption of Lemma 2 is satisfied. Let $A_1 \subset G$ and $Q' \ge Q$. Let $c' \in \alpha^{-1}(\mathcal{L}(\beta(A_1)^{+Q'}))$. Let $d \in \mathcal{L}(A_1)$. Then

$$\begin{aligned}
\alpha(c'd) &\overset{\text{Lemma 2}}{\in} \alpha(c')\,\mathcal{L}(\beta(A_1)^{+Q}) \\
&\subset \alpha(c')\,\mathcal{L}(\beta(A_1)^{+Q'}) \\
&= \mathcal{L}(\beta(A_1)^{+Q'})
\end{aligned}$$

$\Rightarrow\ c'd \in \alpha^{-1}(\mathcal{L}(\beta(A_1)^{+Q'}))$

$\Rightarrow\ \alpha^{-1}(\mathcal{L}(\beta(A_1)^{+Q'}))$ is a union of cosets of $\mathcal{L}(A_1)$. $\square$

Proof of the main theorem: Let

$$\begin{aligned}
\varphi : \mathbb{Z}/n\mathbb{Z} \wr G &\longrightarrow \mathbb{Z}/m\mathbb{Z} \wr H \\
(c, p) &\longmapsto (\alpha(c), \beta(p)).
\end{aligned}$$

let $C, K \geq 0$ be constants s.t. β and $\bar{\beta}$ are (C,K)-Q.I., $\beta \circ \bar{\beta}, \bar{\beta} \circ \beta$ at distance $\leq K$ from identities.

Claim: Let p be a prime and let p_1 (resp. p_2) be the p-valuation of n (resp. of m). Then there is a constant $M \geq 1$ such that

$$\left| |A| - \frac{p_2}{p_1} |\beta(A)^{+K}| \right| \leq M \cdot |\partial_G A|$$

for any $A \subset G$ finite.

Proof of the claim: let $A \subset G$ finite, $B := \bar{\beta}\left(\beta(A)^{+K}\right)^{+K}$. We have

$$A \subset B \subset A^{+(C+3)K}.$$

Any $a \in A$ satisfies $d_G(a, \bar{\beta}(\beta(a))) \leq K$ and $\beta(a) \in \beta(A) \subset \beta(A)^{+K}$. Let $b \in B$. There exists $y \in \beta(A)^{+K}$ such that

$$d_G(b, \bar{\beta}(y)) \leq K.$$

Thus there is $a \in A$, $d_H(y, \beta(a)) \leq K$.

Hence:

$$
\begin{aligned}
d_G(b,a) \leq\ & d_G(b, \bar{\beta}(y)) \\
& + d_G(\bar{\beta}(y), \bar{\beta}(\beta(a))) \\
& + d_G(\bar{\beta}(\beta(a)), a)
\end{aligned}
$$

$$\leq 2K + C \cdot d_H(y, \beta(a)) + K$$

$$\leq (C+3) \cdot K$$

$\Rightarrow \quad B \subset A^{+(C+3)K}$. Thus:

$$
\begin{aligned}
|A| \leq |B| \leq |A^{+(C+3)K}| &= |A| + |A^{+(C+3)K} \setminus A| \\
&\leq |A| + N^{(C+3)K} \cdot |\partial_G A|
\end{aligned}
$$

where $N \geq 3$ is an integer larger than the maximal degree of a vertex in the Cayley graph of G.

On the other hand: $\alpha^{-1}(\chi(\beta(A)^{+K}))$ is a union of cosets of $\chi(A)$ (Lemma 1) thus there is $k \geq 1$ s.t.

$$m^{|\beta(A)^{+k}|} = |\gamma(\beta(A)^{+k})| = |\alpha^{-1}(\gamma(\beta(A)^{+k}))|$$
$$= k \cdot |\gamma(A)| = k \cdot n^{|A|}$$

so the powers of p in these 2 integers are the same:

$$p^{p_2 \cdot |\beta(A)^{+k}|} = E \cdot p^{p_1 \cdot |A|}.$$

Likewise, $\alpha(B)$ is a union of cosets of $\gamma(\beta(A)^{+k})$ (Lemma 1), there is $F \geq 1$ such that:

$$p^{p_1 \cdot |B|} = F \cdot p^{p_2 \cdot |\beta(A)^{+k}|}$$

$$\Rightarrow \quad |B| = \frac{1}{p_1} \log(F) + \frac{p_2}{p_1} \cdot |\beta(A)^{+k}|.$$

Also:

$$\log(F) \leq \log(EF) = p_1 \cdot (|B| - |A|)$$
$$\leq p_1 \cdot N^{(C+3)K} |\partial_G A|$$

hence:

$$\frac{p_2}{p_1}|\beta(A)^{+k}| \leq |B| \leq \frac{p_2}{p_1}|\beta(A)^{+k}| + N^{(C+3)k}|\partial_G A|$$

Finally:

$$\left||A| - \frac{p_2}{p_1}|\beta(A)^{+k}|\right| \leq \big||A| - |B|\big|$$
$$+ \left||B| - \frac{p_2}{p_1}|\beta(A)^{+k}|\right|$$
$$\leq \underbrace{2N^{(C+3)k}}_{=:M}|\partial_G A|. \qquad \underset{\text{Claim}}{\square}$$

Apply the claim with $(A_k)_{k\in\mathbb{N}}$ a Følner sequence of G:

$$\left| 1 - \frac{p_2}{p_1}\frac{|\beta(A_k)^{+k}|}{|A_k|}\right| \leq M \cdot \underbrace{\frac{|\partial_G A_k|}{|A_k|}}_{\xrightarrow[k\to\infty]{} 0}$$

$\Rightarrow \left(\frac{|\beta(A_k)^{+k}|}{|A_k|}\right)_{k\in\mathbb{N}}$ converges to $\frac{p_1}{p_2}$

$\Rightarrow$ there exists a rational $\frac{r}{s}$ such that, for any prime p, $\frac{p_1}{p_2} = \frac{r}{s}$

$\Rightarrow \exists k \geq 1, \quad n = k^r, \quad m = k^s.$

Lastly: Scalingness of β. Let $A \subset H$ be finite. Then:

$$\left| \frac{s}{r} \cdot |A| - |\beta^{-1}(A)| \right| \leq \left| \frac{s}{r} \cdot |A| - \frac{s}{r} \cdot |A^{+k}| \right|$$

$$+ \left| \frac{s}{r} |A^{+k}| - \frac{s}{r} \cdot |\beta(\beta^{-1}(A))^{+k}| \right|$$

$$+ \left| \frac{s}{r} \cdot |\beta(\beta^{-1}(A))^{+k}| - |\beta^{-1}(A)| \right|$$

$$\leq \frac{s}{r} \cdot |A^{+k} \setminus A| + \frac{s}{r} \cdot \left| |A^{+k}| - |\beta(\beta^{-1}(A))^{+k}| \right|$$

$$+ \left| \frac{s}{r} \cdot |\beta(\beta^{-1}(A))^{+k}| - |\beta^{-1}(A)| \right|$$

$$\leq \frac{s}{r} \cdot R \cdot |\partial_H A| + \frac{s}{r} \cdot |A^{+k} \setminus \beta(\beta^{-1}(A))^{+k}|$$

$$+ M \cdot \underbrace{|\partial_G \beta^{-1}(A)|}_{\leq L \cdot |\partial_H A|}$$

One has $A^{+k} \setminus \beta(\beta^{-1}(A))^{+k} \overset{(*)}{\subset} (\partial_H A)^{+(2k-1)}$

$$\Rightarrow |A^{+k} \setminus \beta(\beta^{-1}(A))^{+k}| \leq |(\partial_H A)^{+(2k-1)}|$$

$$\leq p^{2k-1} \cdot |\partial_H A|$$

where $P \geq 3$ is larger than the maximal degree of a vertex in the Cayley graph of H.

All this together:

$$\left| \frac{s}{r} \cdot |A| - |\beta^{-1}(A)| \right| \leq \left(\frac{s}{r} \cdot R + M \cdot L + \frac{s}{r} \cdot P^{2K-1} \right) \cdot |\partial_H A|$$

$\Rightarrow$ β is quasi-$\frac{s}{r}$-to-one.

$\Rightarrow$ φ is quasi-$\frac{s}{r}$-to-one. $\square$

(*) let $y \in A^{+K} \setminus \beta(\beta^{-1}(A))^{+K}$

- $y \notin A$: $\quad y \in A^{+K} \setminus A \subset (\partial_H A)^{+(K-1)}$

$$\subset (\partial_H A)^{+(2K-1)}$$

- $y \in A$: $\quad \exists x \in G,\ d_H(y, \beta(x)) \leq K$

$$y \notin \beta(\beta^{-1}(A))^{+K} \Rightarrow x \notin \beta^{-1}(A)$$

$$\Rightarrow \beta(x) \notin A$$

$$\Rightarrow \beta(x) \in A^{+K} \setminus A \subset (\partial_H A)^{+(K-1)}$$

$$\Rightarrow y \in (\partial_H A)^{+(2K-1)}. \quad \square_{(*)}$$

Theorem (D., 2025): Let N, M be polynomial growth groups, with growth degrees n, m. Let G, H be amenable, finitely presented and in the class $\mathcal{M}_{exp}$ ($G = BS(1,n)$, $n \geq 2$, or $SOL(\mathbb{Z})$, ...).
Then any Q.I. $N \wr G \to M \wr H$ is quasi-$\frac{m}{n}$-to-one.
In particular, any Q.I. $N \wr G \to N \wr G$ is at bounded distance from a bijection, and $Sc(N \wr G) = \{1\}$.

Proposition: $n, m \geq 2$, N_1, N_2 polynomial growth groups of growth degrees n_1, n_2.
$G, H \in \mathcal{M}_{exp}$ finitely presented, amenable.
If $\exists$ a Q.I:

$$\mathbb{Z}/n\mathbb{Z} \wr (N_1 \wr G) \longrightarrow \mathbb{Z}/m\mathbb{Z} \wr (N_2 \wr H)$$

then there are integers $a, r, s \geq 1$

such that $n = a^r$, $m = a^s$

and $\frac{s}{r} = \frac{n_2}{n_1}$.

Proof: If $\varphi : \mathbb{Z}/n\mathbb{Z} \wr (N_1 \wr G) \to \mathbb{Z}/m\mathbb{Z} \wr (N_2 \wr H)$

then $(c, p) \mapsto (\alpha(c), \beta(p))$

and $\beta : N_1 \wr G \to N_2 \wr H$.

Previous thm $\Rightarrow$ β is quasi-$\frac{n_2}{n_1}$-to-one

$\underset{\text{last week}}{\Rightarrow}$ φ is quasi-$\frac{n_2}{n_1}$-to-one.

But Genevois-Tessera's thm

$\Rightarrow$ $\exists a, r, s$, $n = a^r$, $m = a^s$

β is quasi-$\frac{s}{r}$-to-one, and φ also.

$\underset{\text{uniqueness of scaling factor}}{\Rightarrow}$ $\frac{s}{r} = \frac{n_2}{n_1}$. $\square$

Example: $n \geq 2$

$\Rightarrow$ $\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}^2 \wr BS(1,n))$

$\not\sim_{Q.I.}$ $\mathbb{Z}/4\mathbb{Z} \wr (\mathbb{Z}^3 \wr BS(1,n))$